Three short tutorials : *Constraint programming, Linear programming and Knowledge-based systems*.

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IMARA Team – July 8, 2013

PART 1 ---
CONSTRAINT PROGRAMMING

Introduction

- ***Constraint programming is one paradigm for modelling and solving combinatorial problems.***
- Other algorithms: Genetic algorithms, mixed integer programming, search algorithms in a state space (e.g., A*), simulated annealing, tabu search, …
- A combinatorial problem is a problem defined by entities maintaining relations, to which one combination (a solution) must be found.
- One solution, several solutions, best solution.

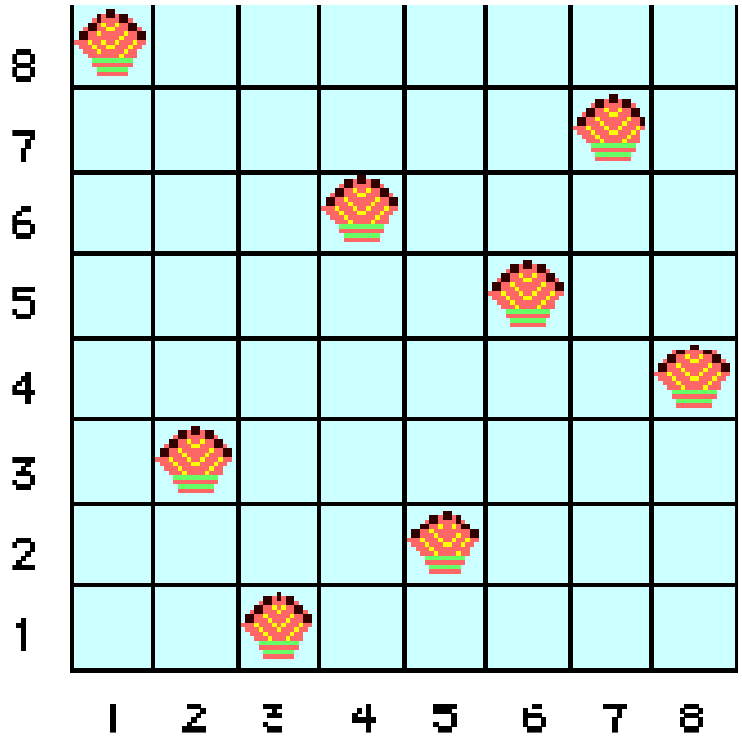
Example (1/3)

- The sudoku game:

		7	8				1	9
8							7	5
4					9			
			5		2	7		
		2	3			1		
5	6			1		3		4
2					6			1
	8	3		2		9	4	7
		5	4	9		8	6	

Example (2/3)

- N-queen problem (here, N = 8):



Example (3/3)

- Cryptarithmetics:

```
  SEND
+ MORE
--------------
 MONEY
```

```
    UN
+ DEUX
+ DEUX
+ DEUX
+ DEUX
------------
  NEUF
```

Model

- Variables with finite domain:
  - *V*ariables : V_i
  - Domains : $D_j = \{ v_1, v_2, \ldots, v_{f(j)} \}$.
  - Forall i, $V_i \in D_i$
- Constraints:
  - For k from 1 to m, $C_k = (X_k, R_k)$ avec :
    - $X_k = \{ V_{i1}, V_{i2}, \ldots, V_{ik} \}$ // The variables involved in constraint C_k
    - $R_k \subset D_{i1} \times D_{i2} \times \ldots \times D_{ik}$ // Possible values of these variables, // together with constraint C_k
- A solution to a CSP (Constraint Satisaction Problem) is a total consistant assignment.

Model / Example

- SEND + MORE = MONEY
- <u>Variables:</u>
  - S, M ∈ {1, 2, 3, 4, 5, 6, 7, 8, 9}
  - E, N, D, O, R, N, Y ∈ {0, 1, 2, 3, 4, 5, 6, 7, 8, 9}
- <u>Constraints:</u>

  $D + E = Y + 10 * R1$

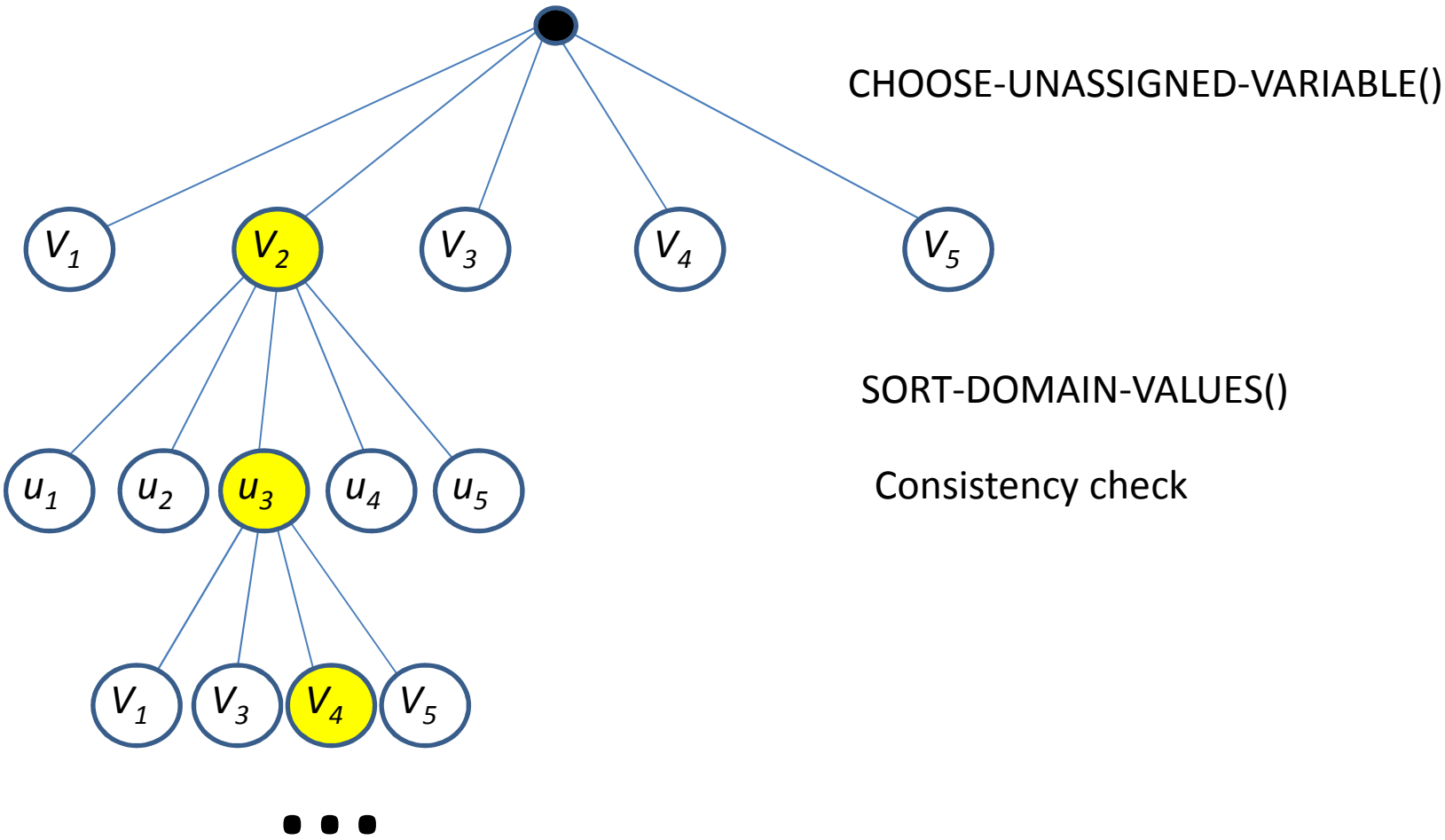
  $N + R + R1 = E + 10 * R2$

  $E + O + R2 = N + 10 * R3$

  $S + M + R3 = O + 10 * M$

  Addition carry variables : R1, R2, R3 ∈ {0, 1}

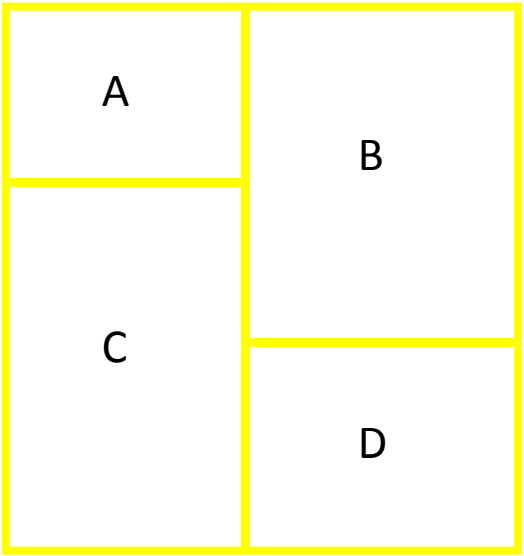
Algorithm



- Backtracking : chronological, backjumping, conflict-directed

Filtering using Forward-Checking

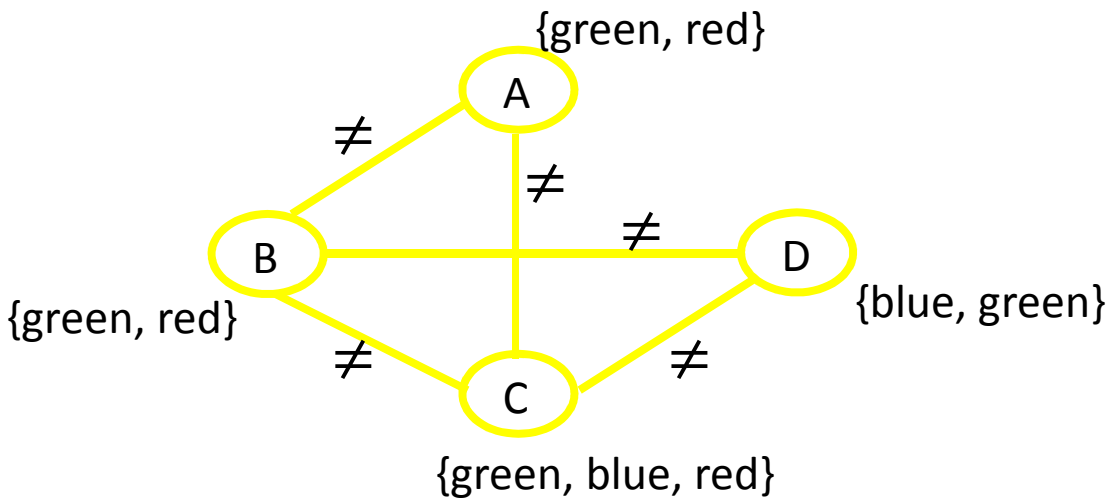
<u>Map coloring:</u> set colors to regions A,B,C and D, while avoiding that two colors are adjacent. Possible colors are :



A and B are green or red
C is green, blue or red
D is blue or green

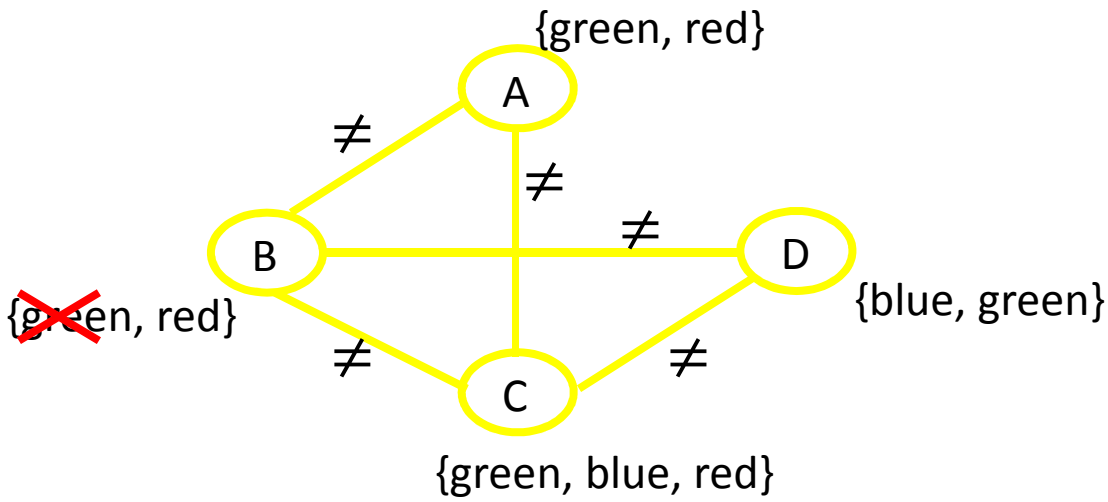
Filtering using Forward-Checking

- Model :



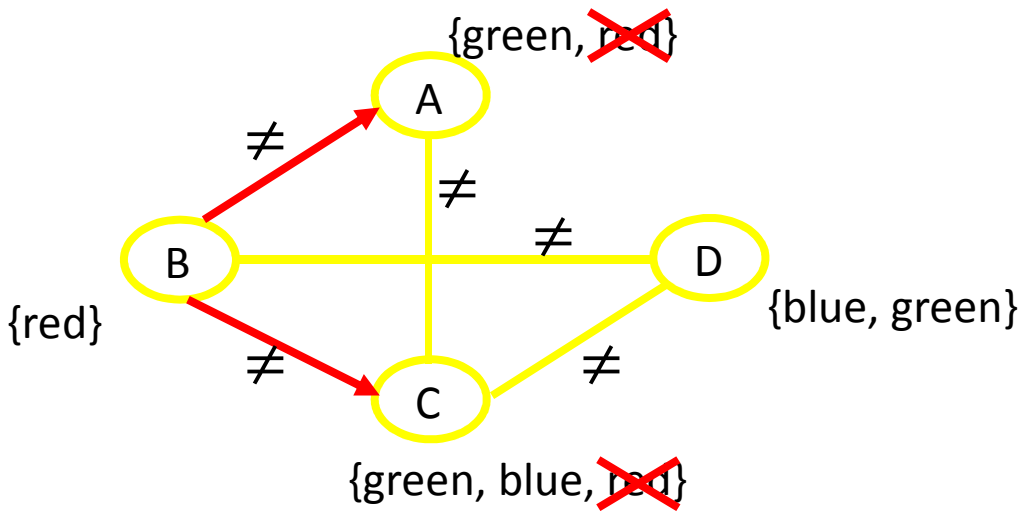
Filtering using Forward-Checking

- Let us assume that B is set to the value RED.



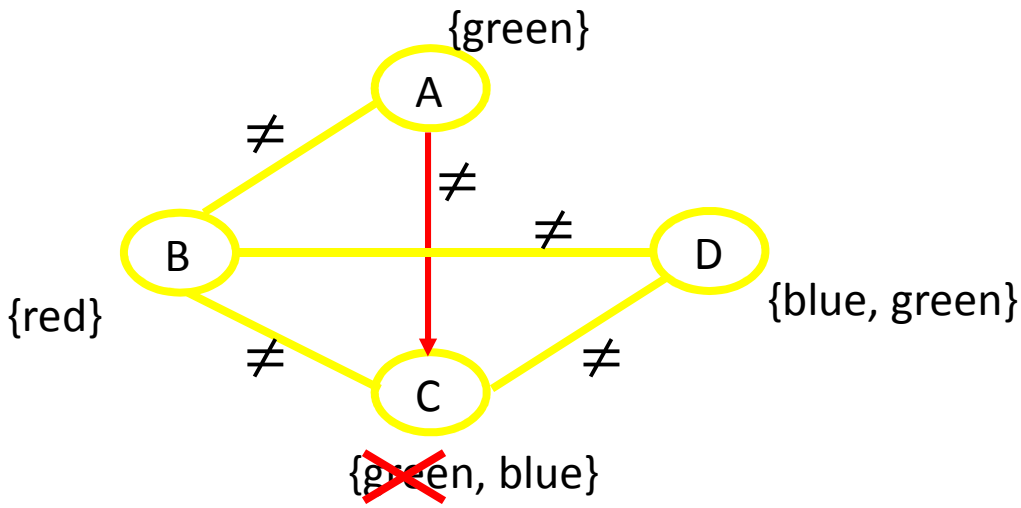
Filtering using Forward-Checking

- Assignment of A to the value GREEN.



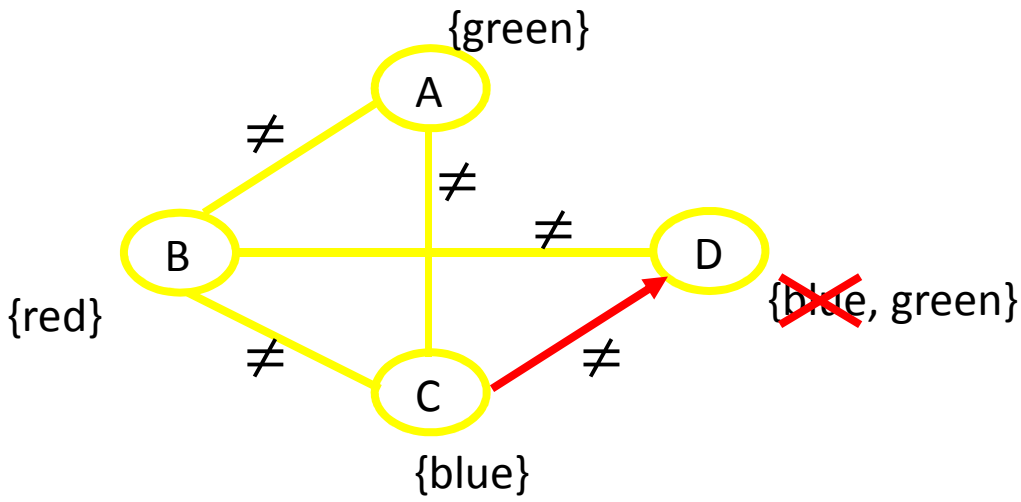
Filtering using Forward-Checking

- Assignment of C to the value BLUE.



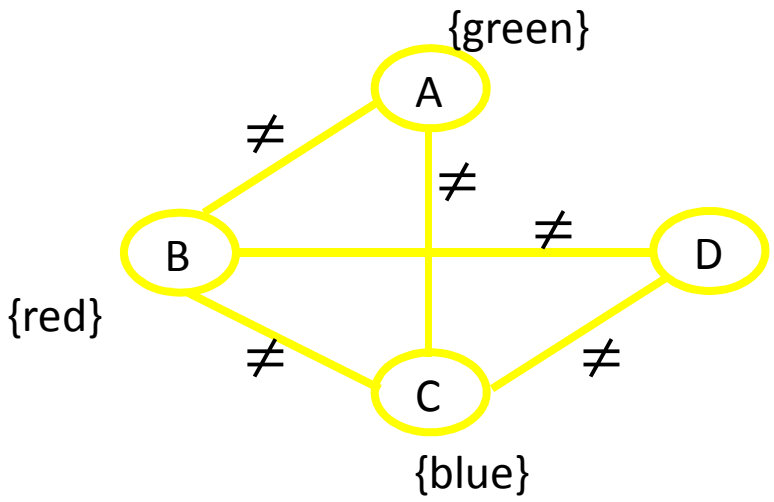
Filtering using Forward-Checking

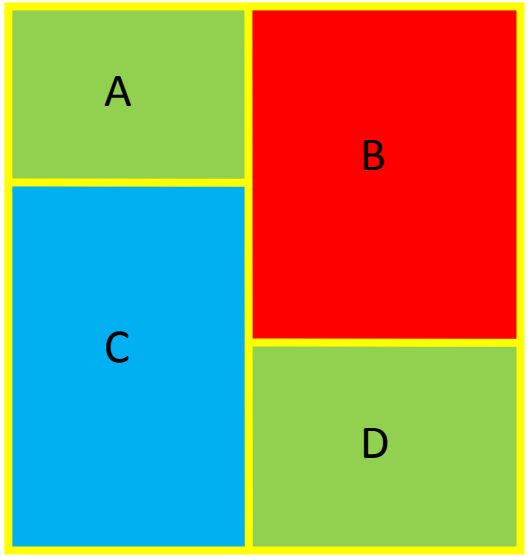
- Assignment of D to the value GREEN.



Filtering using Forward-Checking

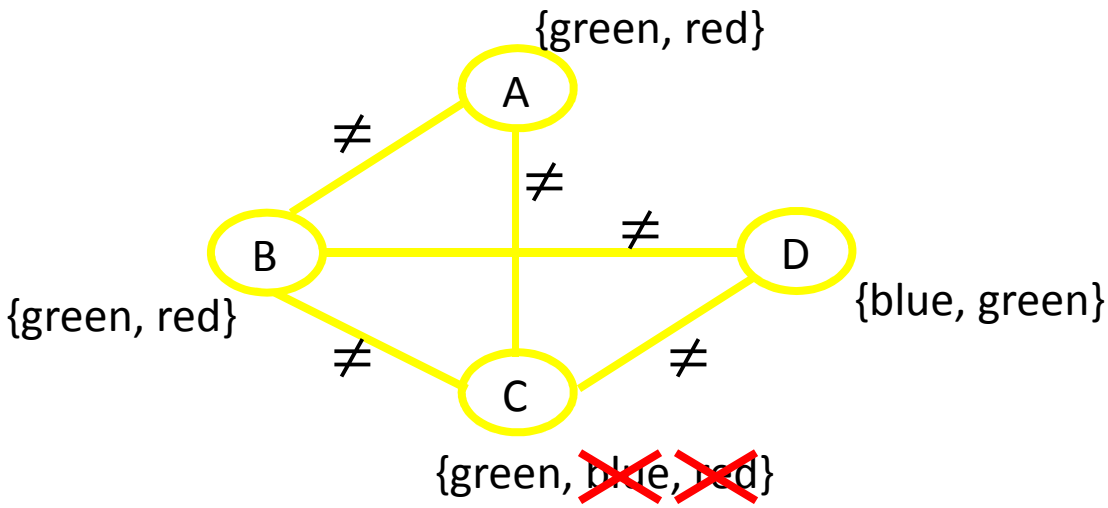
- Solution !





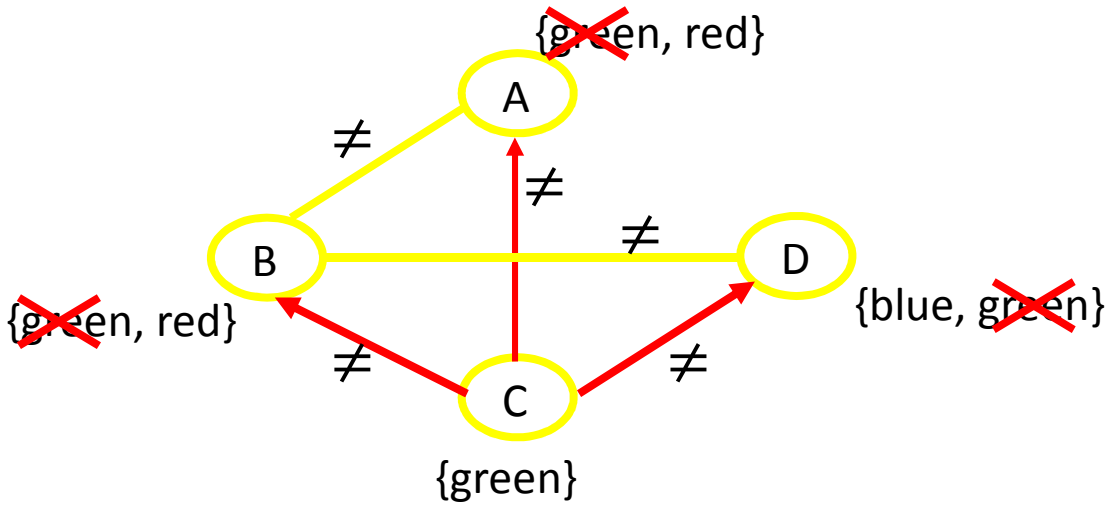
Filtering using Forward-Checking

- But now let us assume that C is set to the value GREEN



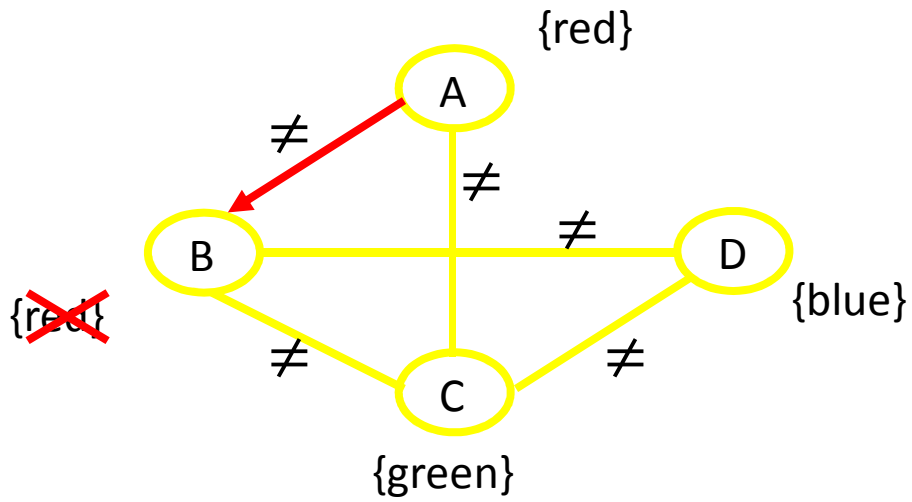
Filtering using Forward-Checking

- Assignment of D to BLUE, and of A and B to RED.



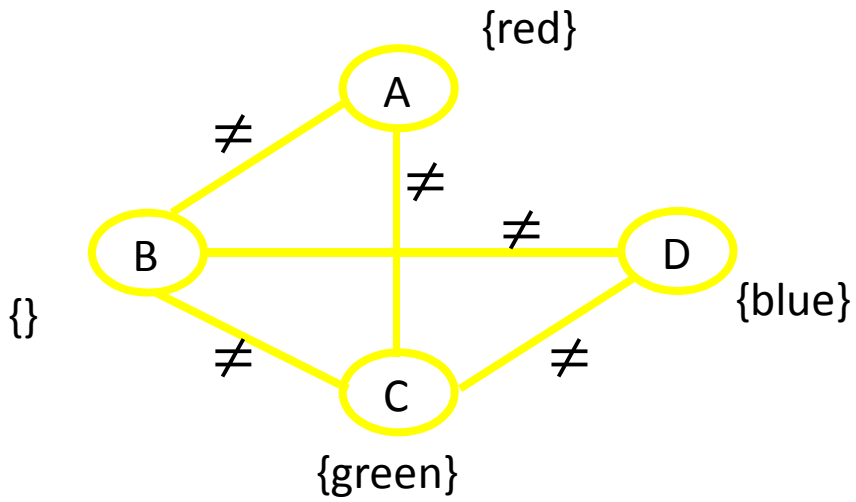
Filtering using Forward-Checking

- Removal of RED from B's domain.



Filtering using Forward-Checking

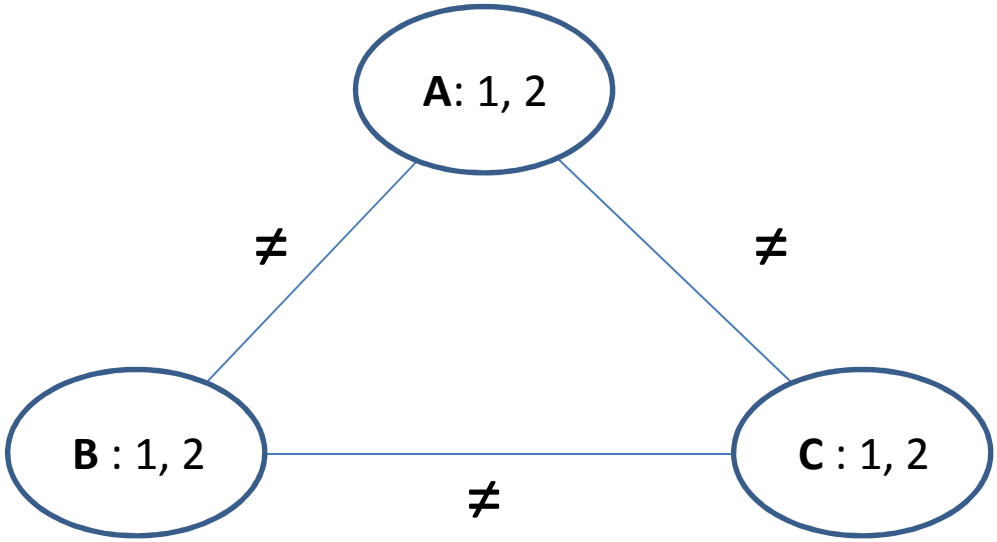
- The domain of B is empty: FAILURE!
  (and then backtracking is required…)



Consistancy

- <u>Definition:</u> A CSP is k-consistant iff, forall subset of $(k-1)$ variables, and forall assignment of these variables, a consistant value can always be assigned to the k-th variable.
- <u>Example:</u>
  - 1-consistancy = node consistancy
  - 2-consistancy = arc consistancy
  - 3-consistancy = path consistancy

Consistancy



- Arc-consistancy is not enough : path consistancy is required, e.g., above.
- Strong k-consistancy.
- Algorithms AC1 to AC8.

Conclusion – Part 1

- **Constraint programming is a paradigm for modelling and solving combinatorial problems**.
- Seminal paper: Jean-Louis Laurière. *A Language and a Program for Stating and Solving Combinatorial Problems*. Artificial Intelligence 10(1): 29-127 (1978).
- An introduction: S. Russel, P. Norvig. *Artificial Intelligence: A Modern Approach*. Prentice Hall, NJ, 2003. Chapter 5.
- OPL Studio from IBM (ex-ILOG):

  http://www-03.ibm.com/software/products/us/en/ibmilogcpleoptistud/
- French Association for Constraint Programming:

  http://afpc.greyc.fr/web/

PART 2 ---
LINEAR PROGRAMMING

Model

- <u>Algebraic notation:</u> n variables, m inequalities.

Min ($\sum_j c_j x_j$) such that

$\forall i, \ \sum_j a_{i,j} xj \geq bi$ // ≤ when Max

$\forall j, \ x_j \geq 0$

- <u>Matrix notation:</u>

Min ${}^tC X$ such that

$A X \geq B$

$X \geq 0$

The Simplex Algorithm: Example

- A factory can build 3 products with production rates of 50, 25 and 75 units/hour, respectively.
- Machines can be used for 45 hours at most.
- Sales benefits of products are 4€, 12€ and 3€, respectively.
- Customers ask for 1000, 500 and 1500 units, respectively.

Example: Model (1/2)

- <u>Variables:</u>

  x_i = number of units of product i built by the factory.

  $i = 1, 2, 3$ et $x_i \geq 0$

- <u>Objective function:</u>

  $z = \max\ (4x_1 + 12x_2 + 3x_3)$

- <u>Constraints:</u>

$$\begin{cases} \frac{x_1}{50} + \frac{x_2}{25} + \frac{x_3}{75} \leq 45 \\ x_1 \leq 1000 \\ x_2 \leq 500 \\ x_3 \leq 1500 \\ x_1, x_2, x_3 \geq 0 \end{cases}$$

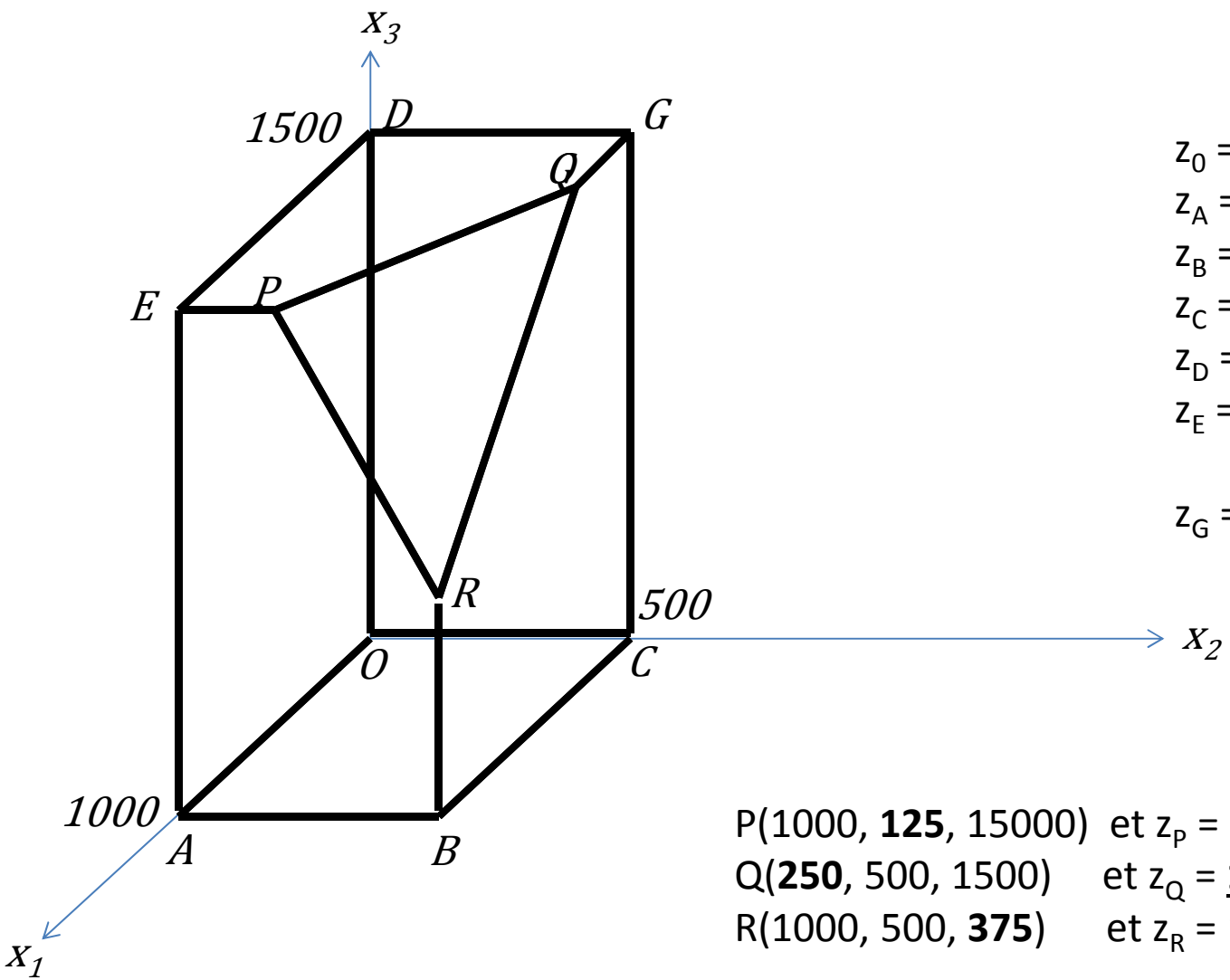
Example: Model (2/2)

- <u>Linear program:</u>

Max ($4x_1 + 12x_2 + 3x_3$) such that

$$\begin{cases} x_1 \leq 1000 & (1) \\ x_2 \leq 500 & (2) \\ x_3 \leq 1500 & (3) \\ 3x_1 + 6x_2 + 2x_3 \leq 6750 & (4) \\ x_1, x_2, x_3 \geq 0 & \end{cases}$$

Example: Geometric model

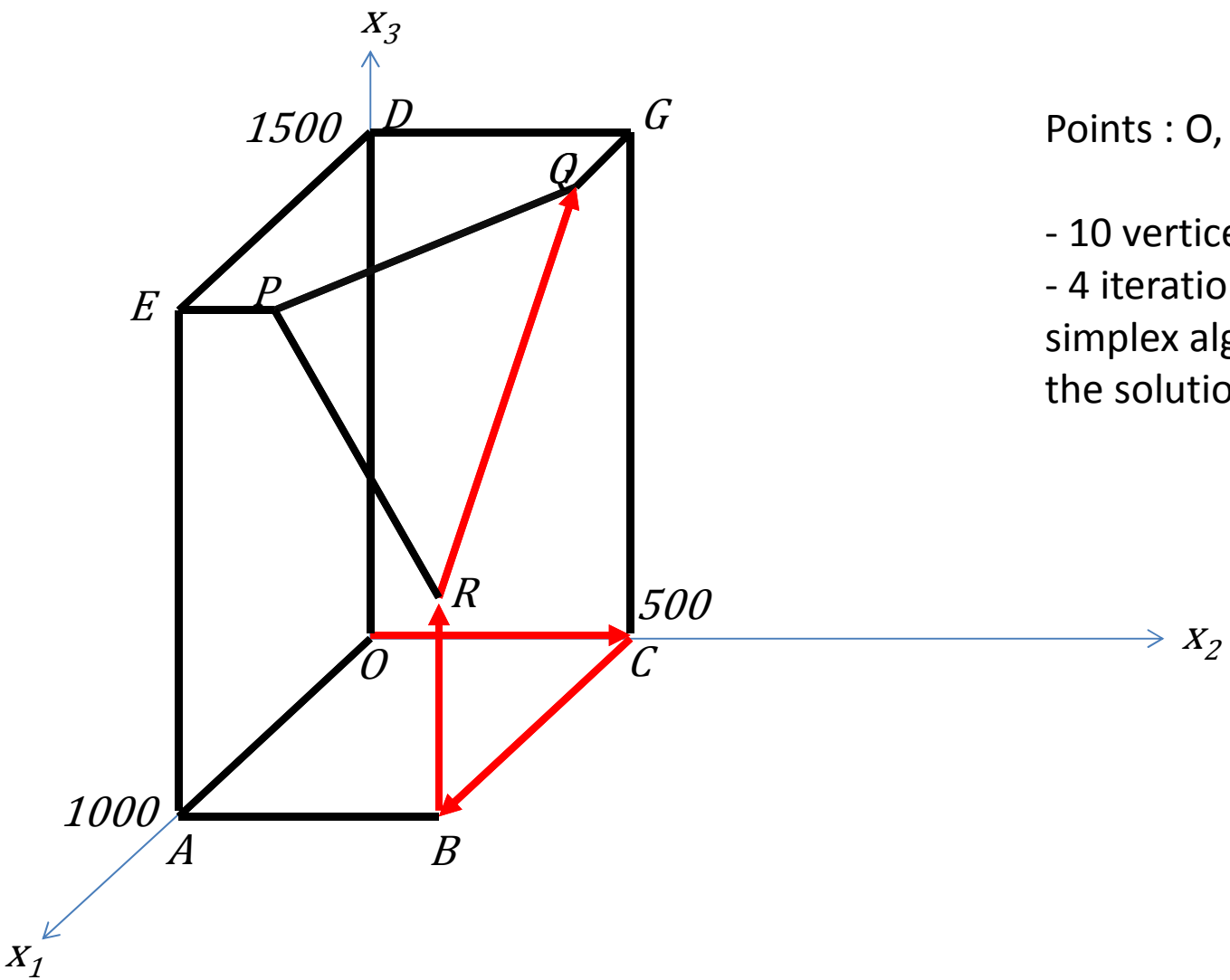


$z_0 = 0$
$z_A = 4000$
$z_B = 7000$
$z_C = 6000$
$z_D = 4500$
$z_E = 8500$

$z_G = 10500$

P(1000, **125**, 15000) et $z_P = 10000$
Q(**250**, 500, 1500) et $z_Q =$ **<u>11500</u>**
R(1000, 500, **375**) et $z_R = 10750$

Example: The simplex algorithm



Points : O, C, B, R, Q.

- 10 vertices to explore
- 4 iterations for the simplex algorithm to find the solution.

Mixed integer programming

- The simplex algorithm finds a solution in R
- What if the variable are **<u>integers</u>**?

  $(x_1, x_2, \ldots, x_n) \in N$ instead of R
- … or are **<u>boolean</u>**? $(x_1, x_2, \ldots, x_n) \in \{0, 1\}$ instead of R

- Branch & bound algorithm.
  - Heuristic search in a tree.
  - A node includes the (relaxed) solution on R and additional constraints.

Example (1/6)

- Let (**P**) be the problem:

max ($4x_1 - x_2$) such that

$$\begin{cases} 7x_1 - 2x_2 \leq 14 \\ x_2 \leq 3 \\ 2x_1 - 2x_2 \leq 3 \\ x_1, x_2 \geq 0 \\ (x_1, x_2) \in N \end{cases}$$

- Let (**P'**) be the problem (**P**) but with variables in R

Branch & bound (2/6)

(P) : $\emptyset$
(P') : $x' = (20/7\,;\,3)$
$z' = 59/7$, donc $z \leq 8$

Branch & bound (3/6)

(P) : $\emptyset$
(P') : $x' = (20/7 ; 3)$
$z' = 59/7$, donc $z \leq 8$

$x_1 \leq 20/7$

(P) : $x_1 \leq 2$
(P') : $x' = (2 ; 1/2)$
$z' = 15/2$, donc $z \leq 7$

$x_1 \geq 20/7$

(P) : $x_1 \geq 3$
(P') : $\emptyset$

Branch & bound (4/6)

(P) : $\emptyset$
(P') : $x' = (20/7 ; 3)$
$z' = 59/7$, donc $z \leq 8$

$x_1 \leq 20/7$

(P) : $x_1 \leq 2$
(P') : $x' = (2 ; 1/2)$
$z' = 15/2$, donc $z \leq 7$

$x_1 \geq 20/7$

(P) : $x_1 \geq 3$
(P') : $\emptyset$

$x_2 \leq 1/2$

(P) : $x_1 \leq 2, x_2 \leq 0$
(P') : $x' = (3/2 ; 0)$
$z' = 6$, donc $z \leq 6$

$x_2 \geq 1/2$

(P) : $x_1 \leq 2, x_2 \geq 1$
(P') : $x' = (2 ; 1)$
$z' = 7$, donc $z \leq 7$

Branch & bound (5/6)

(P) : $\emptyset$
(P') : $x' = (20/7\,;\,3)$
$z' = 59/7$, donc $z \leq 8$

$x_1 \leq 20/7$

(P) : $x_1 \leq 2$
(P') : $x' = (2\,;\,1/2)$
$z' = 15/2$, donc $z \leq 7$

$x_1 \geq 20/7$

(P) : $x_1 \geq 3$
(P') : $\emptyset$

$x_2 \leq 1/2$

(P) : $x_1 \leq 2\,,\, x_2 \leq 0$
(P') : $x' = (3/2\,;\,0)$
$z' = 6$, donc $z \leq 6$

$x_2 \geq 1/2$

(P) : $x_1 \leq 2,\, x_2 \geq 1$
(P') : $x' = (2\,;\,1)$
$z' = 7$, donc $z \leq 7$

$x_1 \leq 3/2$

(P) : $x_1 \leq 1\,,\, x_2 \leq 0$
(P') : $x' = (1\,;\,0)$
$z' = 4$, donc $z \leq 4$

$S_1 = (1\,;\,0\,;\,z = 4)$

$x_1 \geq 3/2$

(P) : $x_1 = 2\,,\, x_2 = 0$
(P') : $\emptyset$

Branch & bound (6/6)

(P) : $\emptyset$
(P') : $x' = (20/7\,;\,3)$
$z' = 59/7$, donc $z \leq 8$

$x_1 \leq 20/7$

(P) : $x_1 \leq 2$
(P') : $x' = (2\,;\,1/2)$
$z' = 15/2$, donc $z \leq 7$

$x_1 \geq 20/7$

(P) : $x_1 \geq 3$
(P') : $\emptyset$

$x_2 \leq 1/2$

(P) : $x_1 \leq 2\,,\, x_2 \leq 0$
(P') : $x' = (3/2\,;\,0)$
$z' = 6$, donc $z \leq 6$

$x_2 \geq 1/2$

(P) : $x_1 \leq 2,\, x_2 \geq 1$
(P') : $x' = (2\,;\,1)$
$z' = 7$, donc $z \leq 7$

$S_2 = (2\,;\,1\,;\,z = 7)$

$x_1 \leq 3/2$

(P) : $x_1 \leq 1\,,\, x_2 \leq 0$
(P') : $x' = (1\,;\,0)$
$z' = 4$, donc $z \leq 4$

$S_1 = (1\,;\,0\,;\,z = 4)$

$x_1 \geq 3/2$

(P) : $x_1 = 2\,,\, x_2 = 0$
(P') : $\emptyset$

Typical problems

- <u>Transportation problem:</u> *Several customers, several depots; A customer can be delivered by several depots; Each depot has a stock capacity; Each customer asks for some quantity; Carrying one unit of product from a depot to a customer has a price.*
- <u>Variants:</u> *… with a multiple of N products; … a customer is assigned to a <u>unique</u> depot; … or a depot can be open/closed with an opening cost;*

Modelling tricks

- Goal programming
- Semi-continuous variable
- IF … THEN …
- Disjunction
- Absolute value (inside cost, inside constraints)
- Quadratic cost
- Step-wise cost

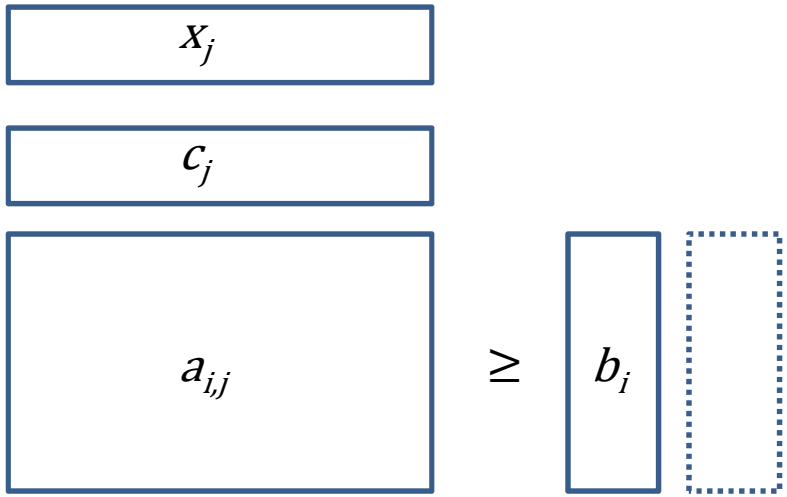
Duality

- <u>Primal lin. prog.:</u>

Min $C\,x$ tel que :

$$A\,x \geq B$$

$$x \geq 0$$

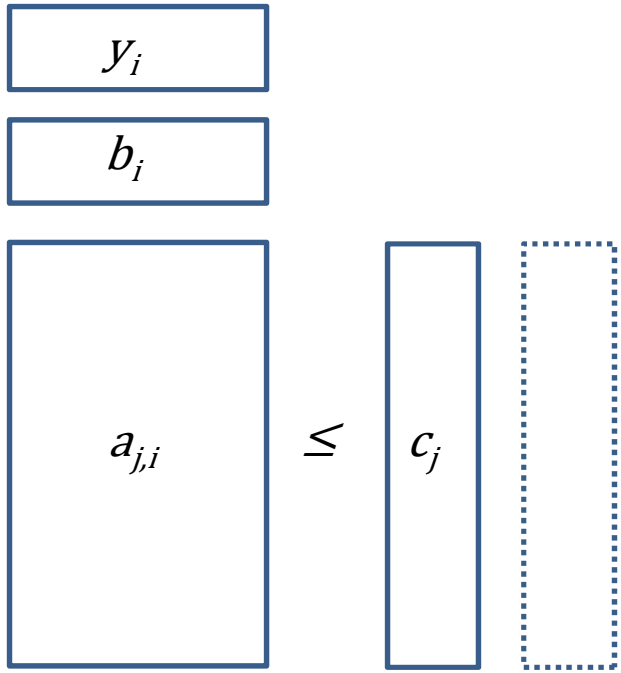


- <u>Dual lin. prog.:</u>

Max ${}^tB\,y$ tel que :

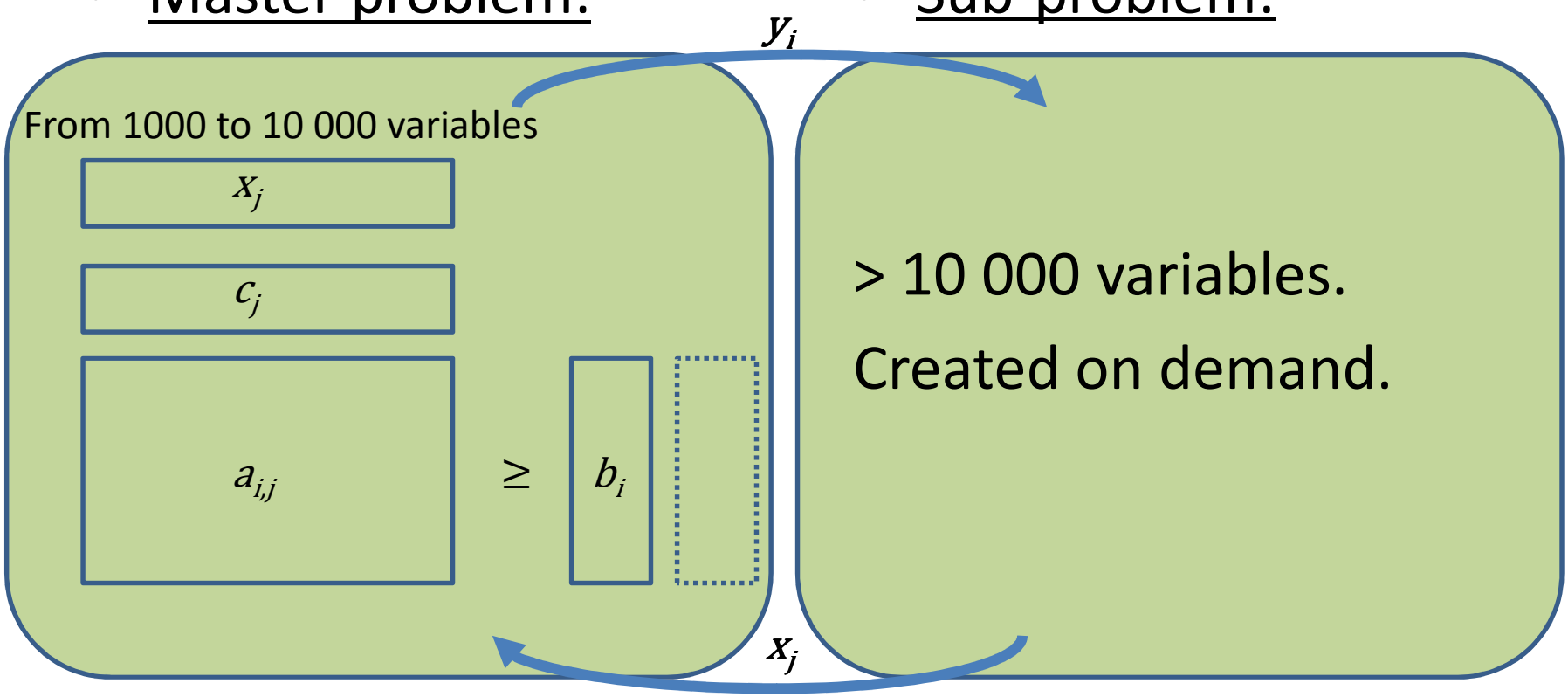
$${}^tA\,y \leq {}^tC$$

$$y \geq 0$$



Column generation algorithm

- <u>Master problem:</u>
- <u>Sub-problem:</u>



WHILE $\exists j / \Delta_j = c_j - \sum_i a_{i,j} y_i < 0$

Multi-criteria optimization

- <u>LP single-criterion:</u>

min ($\sum_j c_j x_j$)

such that

$\forall i, \ \sum_j a_{i,j} x_j \geq b_i$

$\forall j, \ x_j \geq 0$

- <u>LP multi-criteria:</u>

$\forall k$, min ($F_k = \sum_j c_{j,k} x_j$)

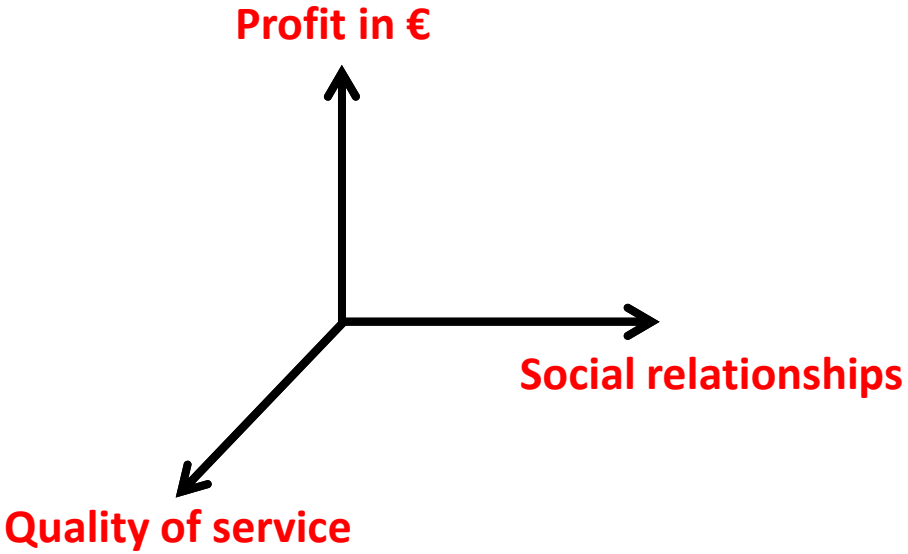
such that

$\forall i, \ \sum_j a_{i,j} x_j \geq b_i$

$\forall j, \ x_j \geq 0$

Multi-criteria optimization: Example

- In a problem of human resource planning, maximize profit, the social relationships and the quality of services.

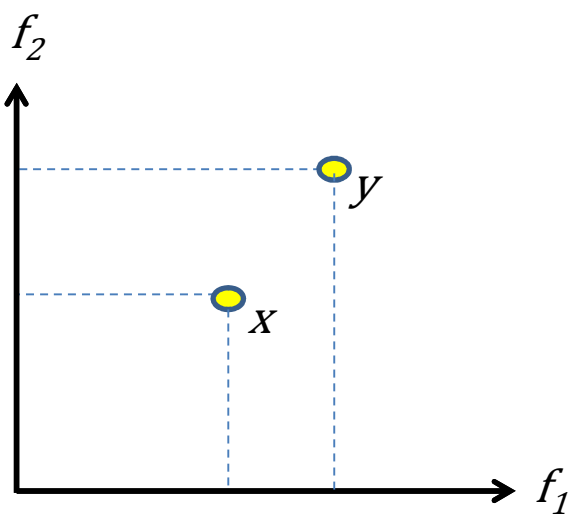


- Criteria are inter-dependent: how to find an « optimal » solution?

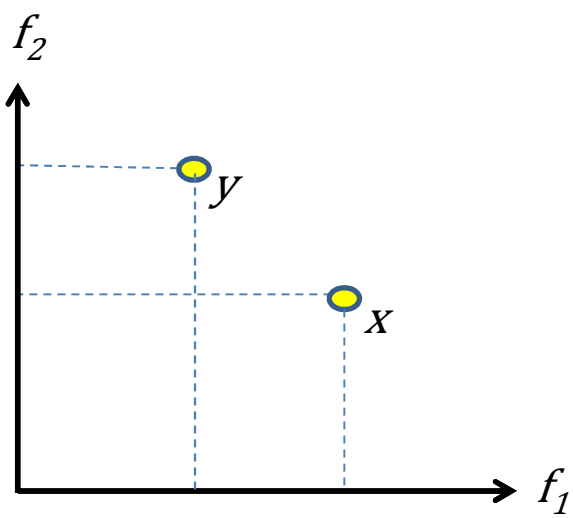
Multi-criteria optimization

- <u>Definition:</u> for a LP of minimization, solution x **dominates** solution y iff

$$x\,D\,y \Leftrightarrow \forall k,\ f_k(x) \leq f_k(y)$$

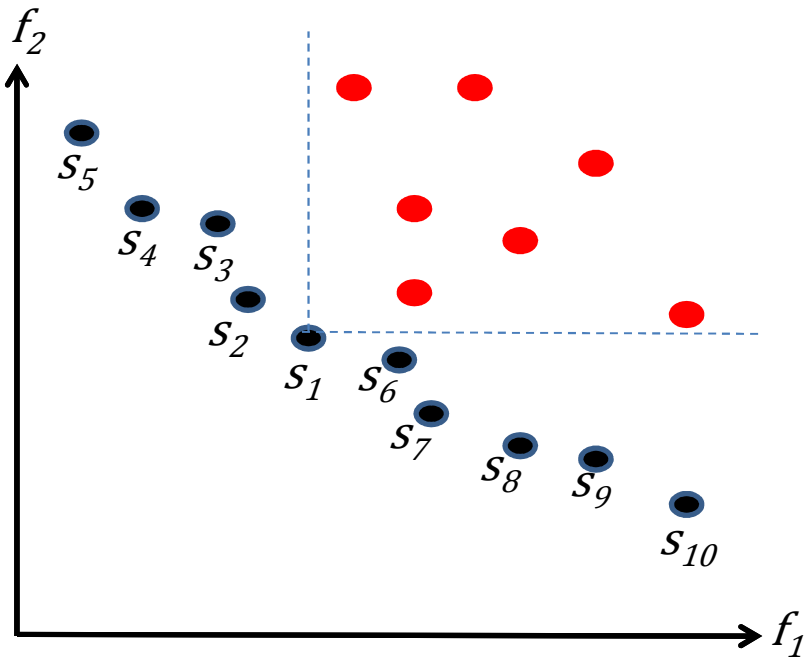


x dominates y



x does **<u>not</u>** dominate y

Multi-criteria optimization



- Upper right quadrant: solutions dominated by s_1.
- Pareto front = { s_1, s_2, …, s_{10} } *// efficient solutions*
- Which solution on the Pareto front is the « best » one?

« Best » solution? (1/2)

- **<u>Criteria aggregation method:</u>**

  $$f(x) = p_1 f_1(x) + p_2 f_2(x) + \ldots + p_n f_n(x)$$

  $$\min f(x)$$

  – Simple but (i) does not necessarily reach an optimum; And (ii) how to set the p_i ?

- **<u>Serialization method:</u>**

  Set an order on criteria,

  and solve the problem along this order.

  – Efficient, problem decomposition, but (i) n optimization to compute; And (ii) the optimum changes along the criterion order.

« Best » solution? (2/2)

- **<u>Gain array method:</u>** Separately solve along every criterion, and minimize the maximum deviation from the ideal point (F_{11}, F_{22}, …, F_{kk}).

Criterion	F_1	F_2	…	F_k
Solution 1	$\mathbf{F_{11}}$	F_{12}	…	F_{1k}
Solution 2	F_{21}	$\mathbf{F_{22}}$	…	F_{2k}
…				
Solution k	F_{k1}	F_{k2}	…	$\mathbf{F_{kk}}$

- Min λ such that

$$\begin{cases} \forall\, k,\ F_k - F_{kk} \leq \lambda\,(\max_i (F_{ik}) - F_{kk}) \\ \forall\, k,\ F_k = \sum_j c_{j,k}\, x_j \\ \forall\, j,\ x_j \geq 0 \end{cases} \qquad // \ 0 \leq \lambda \leq 1$$

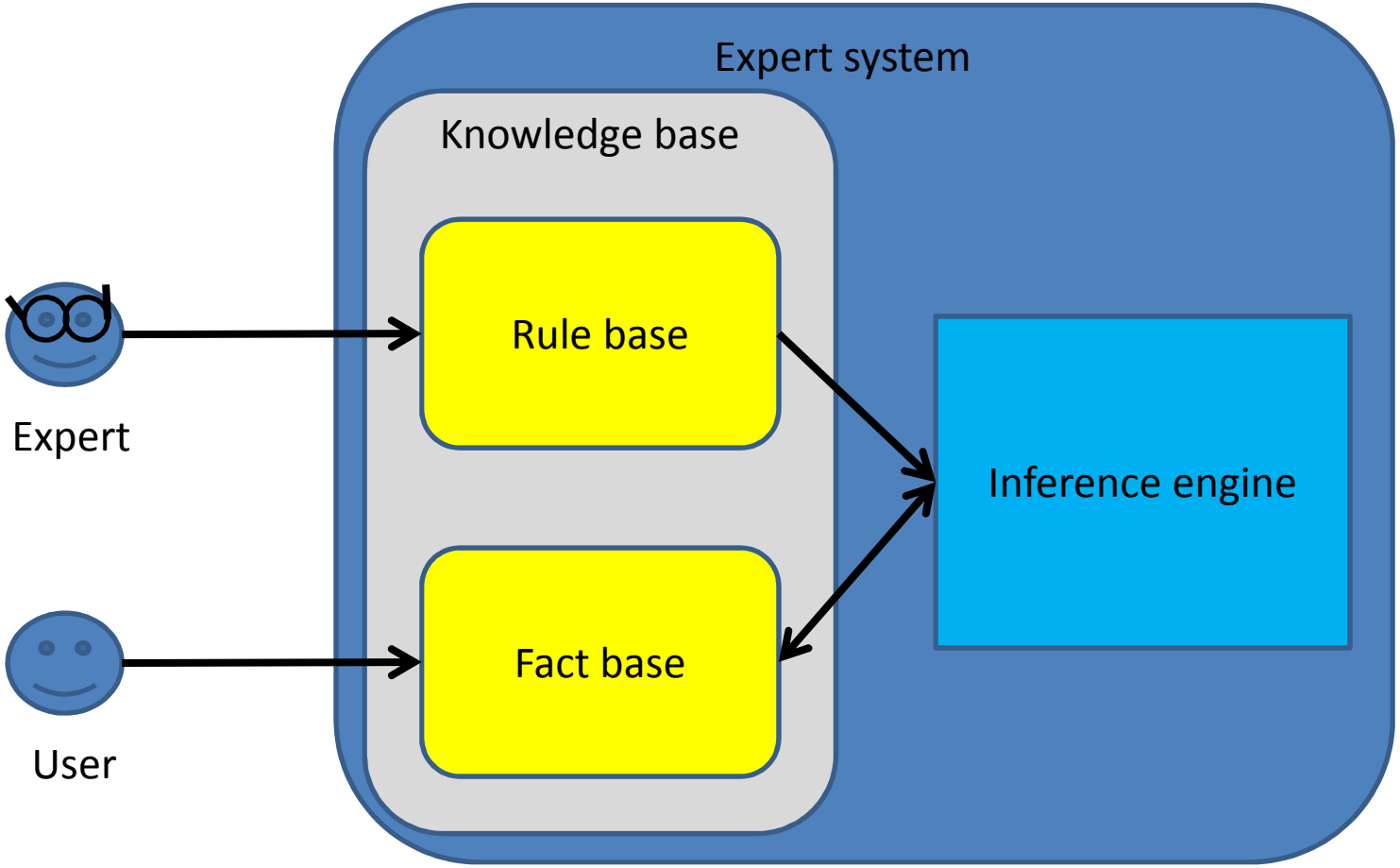
Conclusion – Part 2

- **With the *simplex* algorithm, linear programming finds a solution in R which minimizes a cost function, while respecting linear inequalities.**
- **With the *branch & bound* method, mixed integer programming does the same thing but on N.**
- Seminal paper: G.B. Dantzig. *Maximization of a linear function of variables subject to linear inequalities*, 1947. Published pp. 339–347 in T.C. Koopmans (ed.): *Activity Analysis of Production and Allocation*, New York-London 1951 (Wiley & Chapman-Hall).
- An introduction: E. Jacquet-Lagrèze. *Programmation linéaire. Modélisation et mise en oeuvre informatique*. Economica, 1998.
- CPLEX from IBM (ex-ILOG)

  http://www-03.ibm.com/software/products/us/en/ibmilogcpleoptistud/

PART 3 ---
KNOWLEDGE-BASED SYSTEMS

Expert systems



Inference Rules

- <u>Rule base:</u>

  *(**defrule** is-uncle*

  (<x> ^father-of <y>) (<z> ^sex male) (<z> ^brother-of <x>)

  => { <z> ^uncle-of <y> }

)

- <u>Fact base:</u>

  *(**fact** Louis ^sex male)*

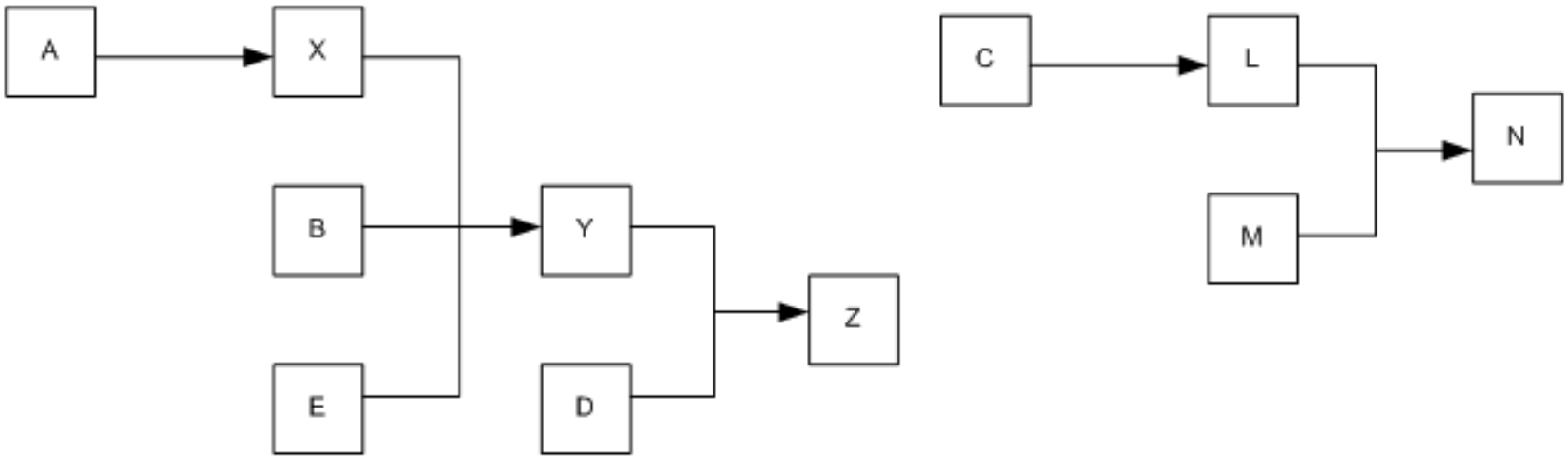
  *(**fact** Louis ^brother-of Alexandre)*

  *(**fact** Alexandre ^father-of Jean)*

- <u>Inferred fact:</u> *(**add-fact** Louis ^uncle-of Jean)*

Forward vs. Backward Chaining

- Rule base:



- Fact base: {A, B, C, D, E, X}
- Questions:
  - Forward: what can be inferred?
  - Backward: can we prove Z? And N?

Inference Engine

- **Procedure FORWARD-CHAINING**(Rules, Facts)

  WHILE there remains rules in Rules to fire

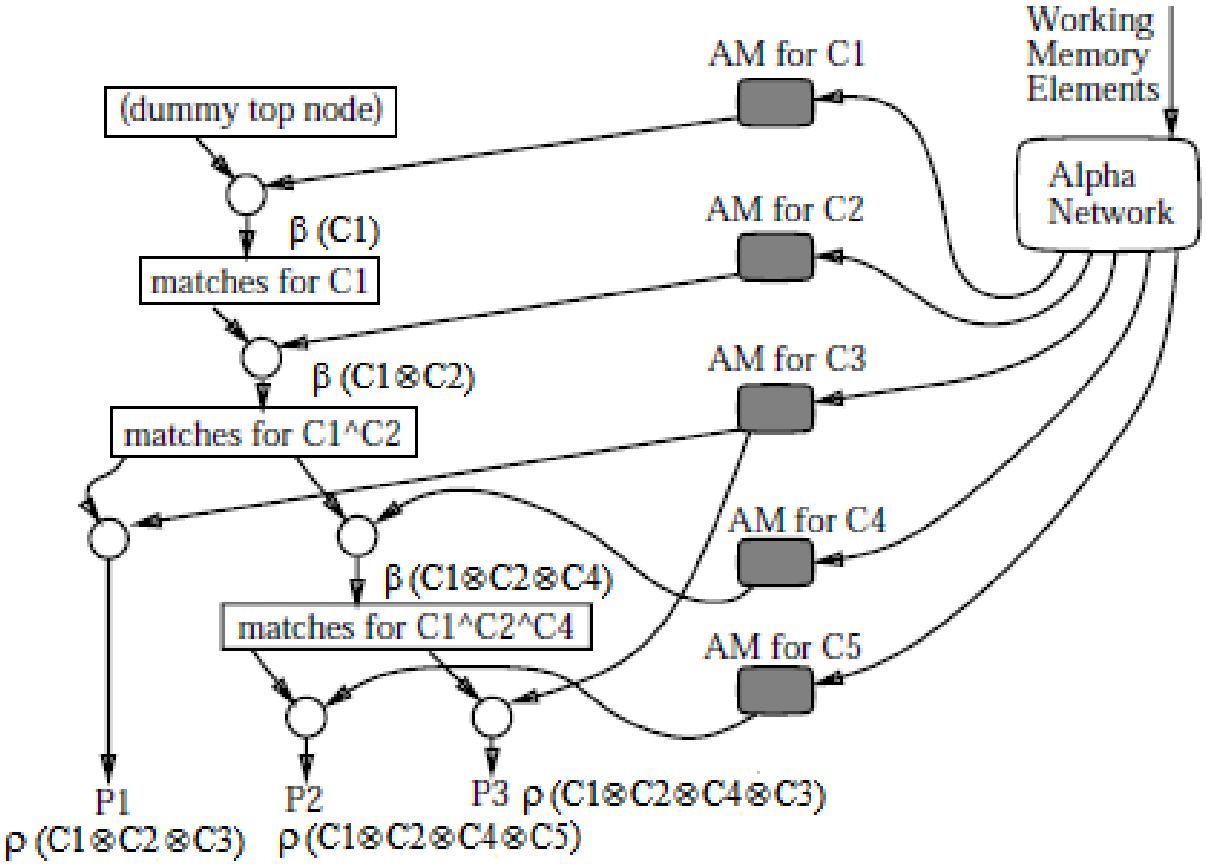
  1. Lookup for rules of Rules which could fire (pattern-matching)
  2. Solve conflicts to choose one candidate rule R
  3. Fire R (inference)

- **Function BACKWARD-CHAINING**(Goal, Rules, Facts)

  1. Lookup for rules of Rules which conclude on Goal
  2. IF one such rule can fire with Facts, THEN fire it. SUCCESS
  3. Solve conflicts to determine one candidate rule R
  4. AND [BACKWARD-CHAINING(LHS(R), Rules, Facts)]

Rete algorithm

- <u>Rule base:</u>
- $P_1 : C_1 \wedge C_2 \wedge C_3$
- $P_2 : C_1 \wedge C_2 \wedge C_4 \wedge C_5$
- $P_3 : C_1 \wedge C_2 \wedge C_4 \wedge C_3$



Conclusion – Part 3

- Knowledge-based system = (rule base + fact base) + inference engine.
- <u>Seminal papers:</u>
  - J. Lederberg, E. Feigenbaum. Mechanization of Inductive Inference in Organic Chemistry. In Formal Representation of Human Judgment, B. Kleinmuntz (ed.), Wiley, 1968. [DENDRAL]
  - B. Buchanan, E. Shortliffe. [MYCIN]
- JESS from Sandia National Laboratories.
  http://herzberg.ca.sandia.gov/

THANK YOU!